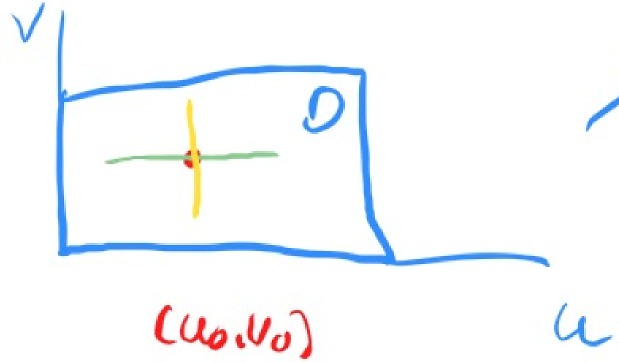
Last time:

Parametrized surfaces:

$$\Phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$



$$T_u = \frac{\partial \Phi}{\partial u}(u_0,v_0) \qquad T_v = \frac{\partial \Phi}{\partial v}(u_0,v_0)$$

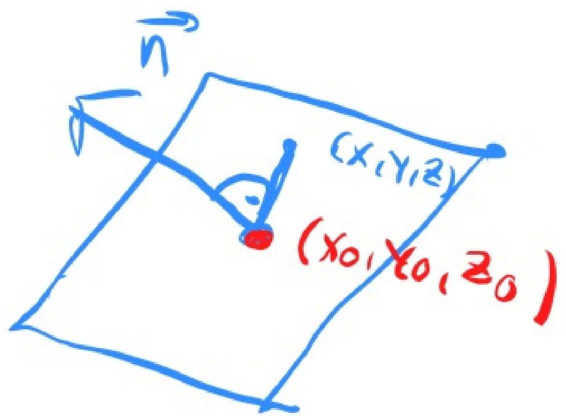
tangent plane = plane spanned by T_u and T_v (if linearly independent!)

Def. A point $\vec{x} = \phi(u,v)$ is called **regular** if $\vec{n} = T_u \times T_v \neq 0$

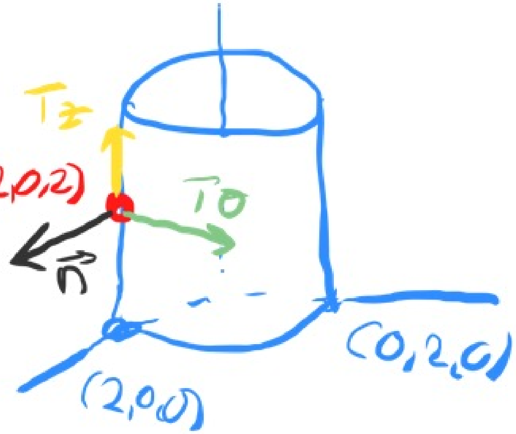
Observe: $\vec{n}$ is perpendicular to tangent plane

$\Longrightarrow$ get equation of tangent plane

$$\boxed{(x-x_0,\, y-y_0,\, z-z_0) \cdot \vec{n} = 0}$$



Example: Recall cylinder of height 3 and radius 2



$\Phi(\theta, z)$

$= (2\cos\theta,\ 2\sin\theta,\ z)$

$T_\theta = \dfrac{\partial\phi}{\partial\theta} = (-2\sin\theta,\ 2\cos\theta,\ 0)$

$T_z = \dfrac{\partial\phi}{\partial z} = (0,\ 0,\ 1)$

$$T_\theta \times T_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

aside

det =

$\vec{i}\begin{vmatrix} 2\cos\theta & 0 \\ 0 & 1 \end{vmatrix}$

$-\vec{j}\begin{vmatrix} -2\sin\theta & 0 \\ 0 & \end{vmatrix}$

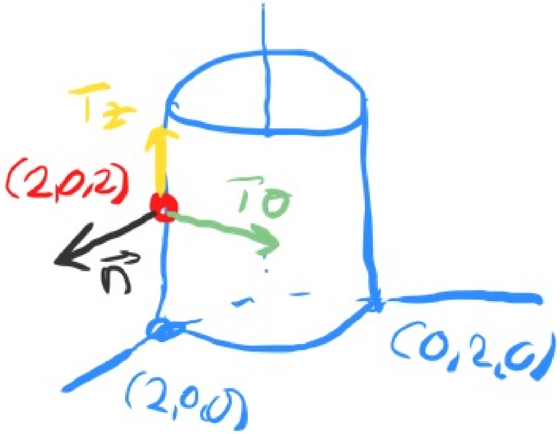
$+\vec{k}\begin{vmatrix} \tilde{0} & \tilde{0} \end{vmatrix}$

$(2\cos\theta,\ 2\sin\theta,\ 0)$

$\parallel$

Example: Recall cylinder of height 3 and radius 2

$(2,0,2) = \Phi(0,2)$



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$- \vec{j} \begin{vmatrix} -2\sin\theta & 0 \\ 0 & \end{vmatrix}$
$+ \vec{k} \begin{vmatrix} \tilde{0} & \tilde{0} \end{vmatrix}$

$$\boxed{\vec{n} = (2\cos\theta,\ 2\sin\theta,\ 0)}$$

$\parallel$

$\Rightarrow \quad \vec{n}(0,2) \quad = (2\cos\theta,\ 2\sin\theta,\ 0)$

$= (2,0,0)$

normal vector at
$\Phi(0,2) = (2,0,2)$

$\Rightarrow$ equation of tangent plane at $(2,0,2)$

$= (x-2,\ y-0,\ z-2) \cdot \vec{n}(0,2)$

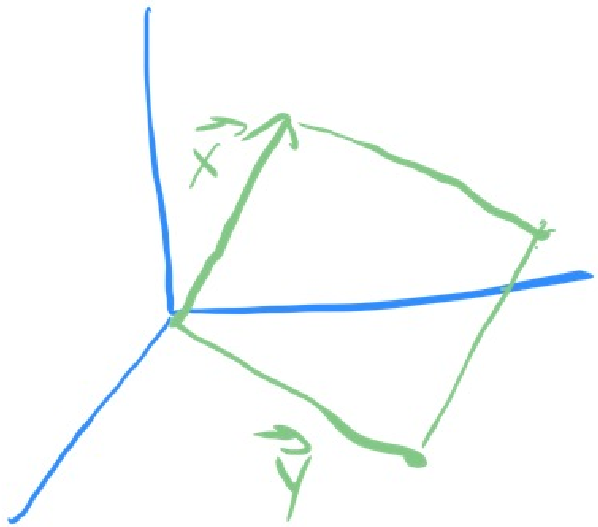
$= (x-2,\ y,\ z-2) \cdot (2,0,0)$

$= 2x-4$

i.e. tangent plane given by $\boxed{x=2}$

7.4 Areas of Surfaces

Consider parallegram spanned by vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$



area of parallelogram $= \| \vec{x} \times \vec{y} \|$

Explicit example:

$$\vec{x} = (1, 2, 3) \qquad \vec{y} = (0, 1, 1)$$

$$\vec{x} \times \vec{y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \vec{i} \underbrace{\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}}_{2-3=-1} - \vec{j} \underbrace{\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}}_{=1} + \vec{k} \underbrace{\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}}_{=1}$$

$$= -\vec{i} - \vec{j} + \vec{k}$$

$$\Rightarrow \|\vec{x} \times \vec{y}\| = \|(-1, -1, 1)\| = \sqrt{(-1)^2 + (-1)^2 + 1^2}$$

$$= \boxed{\sqrt{3}}$$

Problem: Calculate area of a parametrized surface.

<u>Theorem</u> $\Phi : D \to \mathbb{R}^3$ parametrized surve

$$\Rightarrow \boxed{\text{area of surface} = \iint_D \| T_u \times T_v \| \, du \, dv}$$

<u>Remark</u>: we need that Φ is essentially 1-1

essentially 1-1 means

1-1 up to a subset of D of area 0

e.g. for cylinder: $\boxed{\Phi : [0, 2\pi] \times [0,3] \to \mathbb{R}^3}$

recall:

$$\boxed{\begin{aligned}&\Phi(\theta, z) = \\ &(2\cos\theta, 2\sin\theta, z)\end{aligned}}$$

$\Phi(0, z) = \Phi(2\pi, z)$

1-1 only violated for points $(0,z)$ and $(2\pi, z)$

points $\{(0,z), 0 \le z \le 3\}$ and $\{(2\pi, z), 0 \le z \le 3\}$

are lines $\rightarrow$ area of lines $= 0$

<u>Example:</u> Check area formula for cylinder:

$\Phi : [0, 2\pi] \times [0,3] \longrightarrow \mathbb{R}^3$

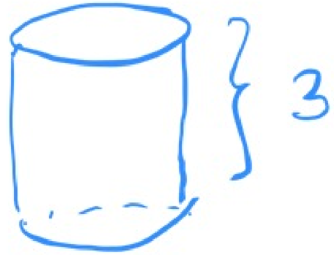
already calculated:

$$T_\theta = (-2\sin\theta, 2\cos\theta, 0)$$
$$T_z = (0, 0, 1)$$

$$\vec{n} = T_\theta \times T_z = (2\cos\theta, 2\sin\theta, 0)$$

$$\| T_\theta \times T_z \| = \sqrt{4\cos^2\theta + 4\sin^2\theta + 0}$$
$$= 2$$

area of cylinder



$$= \iint_D \| T_\theta \times T_z \| \, d\theta \, dz$$

$$= \int_0^3 \int_0^{2\pi} 2 \, d\theta \, dz = 3 \cdot 2\pi \cdot 2$$

known: area of cylinder of radius r and height h

$$= 2\pi r h$$

Example: surface = $z = f(x,y)$ graph of a function

for a function $f: D \subset \mathbb{R}^2 \to \mathbb{R}^3$

choose u,v coordinates for D

$x = u$

$y = v$

$z = f(u,v)$

$$\Phi(u,v) = (u, v, f(u,v))$$

$$\Rightarrow T_u = \frac{\partial \Phi}{\partial u} = \left(1, 0, \frac{\partial f}{\partial u}\right)$$

$$T_v = \frac{\partial \Phi}{\partial v} = \left(0, 1, \frac{\partial f}{\partial v}\right)$$

$$T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial u} \\ 0 & 1 & \frac{\partial f}{\partial v} \end{vmatrix} = \dots = \boxed{\left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right) = T_u \times T_v}$$

$$\Rightarrow \text{area} = \iint_D \|T_u \times T_v\| \, du\,dv$$

$$\boxed{\text{area} = \iint_D \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1} \; du\,dv}$$

example: calculate area of paraboloid

$z = x^2 + y^2$, $\quad z \leq 4$

sol.

$$\phi(u,v) = (u, v, \underbrace{u^2+v^2}_{f(u,v)})$$

$$\Rightarrow \frac{\partial f}{\partial u} = 2u$$

$$\frac{\partial f}{\partial v} = 2v$$

$$\Rightarrow \text{area} = \iint_D \sqrt{4u^2+4v^2+1}\; du\, dv$$

$$D = \{(u,v),\quad 0 \leq z = u^2+v^2 \leq 4\}$$

$$\Rightarrow \quad u^2+v^2 \leq 4$$

disk of radius 2

$\Rightarrow$ can solve integral $\displaystyle\iint_{u^2+v^2\leq 4} \sqrt{4(u^2+v^2)+1}\; du\, dv$

using polar coordinates: $= \frac{\pi}{6}\left(17^{3/2}-1\right)$